# A New Fekete-Szego Inequality Along with Its Subclasses Extremals and Singularities with Classes of Analytic Function

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**Abstract:**

In this Paper we have introduced a new Fekete-Szegö inequality with classes of analytic functions along with its subclasses and extremals by using principle of subordination and as so obtained sharp upper Bound of the function

$\boldsymbol{f(z) = z + \sum_{n=2}^{\infty} a_n z^n}$ belonging to these classes are also obtained.

**Keywords** :Bounded functions,convex function, extremal function, Starlike functions, Univalent functions.

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## 1.Introduction

Let $\mathcal{A}$denotes the class of the function of the form

$$\boldsymbol{f(z) = z + \sum_{n=2}^{\infty} a_n z^n} \tag{1}$$

Which are analytic function in the unit disc$\mathbb{E} = \{z: |z| < 1|\}$ ,

Let $\boldsymbol{S}$ be the class of the functions of the form (1.1) which are analytic univalent in $\mathbb{E}$. BieberBach[7] proved that $|a_2| \leq 2$ for the functions $f(z) \in S$. And Löwner[5] proved that $|a_3| \leq 3$ for the functions $f(z) \in S$..

With the known estimates this inequality plays an important role to determining estimates of higher coefficients for some sub classes of $\boldsymbol{S}$ (Chhichra[11], Babalola[6]). Using Löwner's method[5], Fekete and szego investigated a well known relation between $a_3$ and $a_2^2$ for the class $\boldsymbol{S}$

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & ,if \mu \leq 0 \\ 1 + 2e^{\left(\frac{-2\mu}{1-\mu}\right)} & ,if\ 0 \leq \mu \leq 1 \\ 4\mu - 3 & ,if \mu \geq 1 \end{cases} \tag{2}$$

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The **Fekete–Szegő inequality** is an inequality for the coefficients of univalent analytic functions found by Fekete and Szegő [10], related to the Bieberbach conjecture. Finding similar estimates for other classes of functions is called the **Fekete–Szegő problem**.

Let **S*** be the subclass of $\mathcal{S}$ of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{A}$ satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \quad (3)$$

We are aware that a function $f(z) \in \mathcal{A}$ is said to be close to convex if there exist $g(z) \in$ **S*** such that

$$Re\frac{((zf'(z))}{g(z)} > 0, z \in \mathbb{E}. \quad (4)$$

Kaplan[18] proved that close to convex functions are univalent.

$$S^*(A,B) = \{f(z) \in \mathcal{A};\ \frac{((zf'(z))}{g(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B \le A \le 1, z \in \mathbb{E}\} \quad (5)$$

Where$S^*(A,B)$ is a subclass of $S^*$.

Fekete-Szegö problem was studied by Abedel-Gawad[4] in the context of alpha quasi-convex function. Goel and Mehrok[12], Al-Shaqsi and Darus[9], Hayami and Owa[17], Al-Abbadi and Darus[2]have investigated the upper bound of $|a_3 - \mu a_2^2|$for different functions in the class S.

Gurmeet singh et al.[3] also introduced the class of inverse Starlike functions
g(z) $= z + \sum_{n=2}^{\infty} b_n z^n \in A$ which satisfies

$$Re\left(\frac{zf(z)}{2\int_0^z f(z)dz}\right) > 0,\ z \in E \quad \text{i.e.}\ \frac{zf(z)}{2\int_0^z f(z)dz} < \frac{1+z}{1-z}$$

Gandhi et al.[14]and Rathore et al.[2]established a new class of analytic functions with Fekete-szego inequality using subordination method.
We introduce the class $\mathcal{A}(\alpha,\beta)$ of functions$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ with satisfying the condition

$$\alpha\left[\frac{zf''(z)}{f'(z)}\right] + \beta\left[\frac{zf'(z)}{\{zf(z)\}'}\right] \prec \left(\frac{1+z}{1-z}\right) \quad (6)$$

Let$\mathcal{A}(\alpha,\beta;A,B)$denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$\alpha\left[\frac{zf''(z)}{f'(z)}\right] + \beta\left[\frac{zf'(z)}{\{zf(z)\}'}\right] \prec \left(\frac{1+Az}{1+Bz}\right);\ -1 \le B \le A \le 1 \quad (7)$$

**Let** $\mathcal{A}(\alpha,\beta;\delta)$denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ with satisfying the condition

$$\alpha\left[\frac{zf''(z)}{f'(z)}\right] + \beta\left[\frac{zf'(z)}{\{zf(z)\}'}\right] \prec \left(\frac{1+z}{1-z}\right)^{\delta};\quad \delta > 0 \quad (8)$$

**Let** $\mathcal{A}(\alpha,\beta;A,B,\delta)$denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z)=z+\sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ with satisfying the condition

$$\alpha\left[\frac{zf''(z)}{f'(z)}\right]+\beta\left[\frac{zf'(z)}{\{zf(z)\}'}\right] \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta} - 1 \le B \le A \le 1\,,\delta>0 \qquad (9)$$

Here, Symbol $\prec$ stands for subordination.

**Principle of Subordination :** If $f(z)$ and $F(z)$ are two functions which are analytic in $\mathbb{E}$, then $f(z)$ is called a subordinate to $F(z)$ in $\mathbb{E}$, if there exists a function $w(z)$ which is analytic in $\mathbb{E}$ satisfying the conditions

(i) $w(0)=0$ and (ii) $|w(z)|<1$

such that $f(z)=F\big(w(z)\big)$ , where $z\in\mathbb{E}$ and we denote it as $f(z)\prec F(z)$.

Let $\mathcal{U}$ denote the class of analytic bounded functions of the form

$w(z)=\sum_{n=1}^{\infty} d_n z^n\,, w(0)=0, |w(z)|<1$

Having the restrictions$|d_1|\le 1, |d_2|\le 1-|d_1|^2$.

## 2. Main Results :

**THEOREM 1. :** Prove that

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{4\alpha+\beta+2}{(4\alpha+\beta)(3\alpha+\beta)}-\dfrac{4\mu}{(4\alpha+\beta)^2} & ,if\mu\le\dfrac{(4\alpha+\beta)}{2(3\alpha+\beta)} \qquad (10)\\ \dfrac{1}{3\alpha+\beta} & ,if\,\dfrac{4\alpha+\beta}{2(3\alpha+\beta)}\le\mu\le\dfrac{(4\alpha+\beta+1)(4\alpha+\beta)}{2(3\alpha+\beta)} \qquad (11)\\ \dfrac{4\mu}{(4\alpha+\beta)^2}-\dfrac{(4\alpha+\beta+2)}{(4\alpha+\beta)(3\alpha+\beta)} & ,if\mu\ge\dfrac{(4\alpha+\beta+1)(4\alpha+\beta)}{2(3\alpha+\beta)} \qquad (12)\end{cases}$$

And the results aresharp.

**Proof:**

On Expanding (6) we have

$$\beta+(4a_2\alpha+a_2\beta)z+(6a_3\alpha+2a_3\beta-4a_2^2\alpha-a_2^2\beta)z^2 \prec 1+2c_1z+2(c_2+c_1^2)z^2+\ldots \quad (13)$$

After identifying the terms in (13), we have

$$|a_3-\mu a_2^2| \le \left|\frac{1}{(3\alpha+\beta)}\left\{c_2+c_1^2+\frac{2c_1^2}{4\alpha+\beta}\right\}-\mu\frac{4c_1^2}{(4\alpha+\beta)^2}\right|$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{1}{(3\alpha+\beta)} + \left[\left|\frac{4\alpha+\beta+2}{(4\alpha+\beta)(3\alpha+\beta)} - \frac{4\mu}{(4\alpha+\beta)^2}\right| - \frac{1}{(3\alpha+\beta)}\right]|c_1|^2 \quad (14)$$

**Case I :**If , $\mu \leq \frac{(4\alpha+\beta+2)(4\alpha+\beta)}{4(3\alpha+\beta)}$,then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(3\alpha+\beta)} + \left[\frac{2}{(4\alpha+\beta)(3\alpha+\beta)} - \frac{4\mu}{(4\alpha+\beta)^2}\right]|c_1|^2 \quad (15)$$

**Subcase I(a) :** If, $\mu \leq \frac{4\alpha+\beta}{2(3\alpha+\beta)}$,then

$$|a_3 - \mu a_2^2| \leq \frac{4\alpha+\beta+2}{(4\alpha+\beta)(3\alpha+\beta)} - \frac{4\mu}{(4\alpha+\beta)^2} \quad (16)$$

**Subcase I(b) :** If , $\mu \geq \frac{4\alpha+\beta}{2(3\alpha+\beta)}$,then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(3\alpha+\beta)} \quad (17)$$

**Case II :** If , $\mu \geq \frac{(4\alpha+\beta+2)(4\alpha+\beta)}{4(3\alpha+\beta)}$,then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(3\alpha+\beta)} + \left\{\frac{4\mu}{(4\alpha+\beta)^2} - \frac{8\alpha+\beta+2}{(4\alpha+\beta)(3\alpha+\beta)}\right\} \quad (18)$$

**Subcase II(a) :** If , $\mu \leq \frac{(4\alpha+\beta+1)(4\alpha+\beta)}{\mathbf{2}(3\alpha+\beta)}$,then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(3\alpha+\beta)} \quad (19)$$

***Subcase II(b) :*** *If ,* $\mu \geq \frac{(4\alpha+\beta+1)(4\alpha+\beta)}{\mathbf{2}(3\alpha+\beta)}$,*then*

$$|a_3 - \mu a_2^2| \leq \frac{4\mu}{(4\alpha+\beta)^2} - \frac{4\alpha+\beta+2}{(4\alpha+\beta)(3\alpha+\beta)} \quad (20)$$

Subcase II(a) and I(b) gives the common result.

$$|a_3 - \mu a_2^2| \leq \frac{1}{(3\alpha+\beta)} \quad (21)$$

Under the restriction

$$\frac{4\alpha+\beta}{2(3\alpha+\beta)} \leq \mu \leq \frac{(4\alpha+\beta+1)(4\alpha+\beta)}{\mathbf{2}(3\alpha+\beta)}$$

this complets the theorem, hence the results are sharp.

**Extremal function**

Extreme value for first and third function is $z\{1 + pz\}^{Q}$ (22)

where $P = \frac{2(3\alpha+\beta)-(4\alpha+\beta)(4\alpha+\beta+2)}{(4\alpha+\beta)(3\alpha+\beta)}$ , $Q = \frac{2(3\alpha+\beta)}{2(3\alpha+\beta)-(4\alpha+\beta)(4\alpha+\beta+2)}$

Extreme value for second function is $\frac{z}{(1-z^2)^{\frac{1}{(3\alpha+\beta)}}}$ (23)

**THEOREM 2. :** Prove that

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{(A-B)}{2(3\alpha+\beta)}\left[\frac{A-B-B(4\alpha+\beta)}{(4\alpha+\beta)}\right] - \frac{(A-B)^2\mu}{(4\alpha+\beta)^2}, if \mu \leq \left[\frac{(A-B)-B(4\alpha+\beta)-4\alpha-\beta}{2(A-B)(3\alpha+\beta)}\right](4\alpha+\beta) & (24) \\ \frac{A-B}{3\alpha+\beta}, if \frac{(4\alpha+\beta)[(A-B)-B(4\alpha+\beta)-4\alpha-\beta]}{2(3\alpha+\beta)(A-B)} \leq \mu \leq \frac{(4\alpha+\beta)[(A-B)-B(4\alpha+\beta)+4\alpha+\beta]}{2(3\alpha+\beta)(A-B)} & (25) \\ \frac{(A-B)^2\mu}{(4\alpha+\beta)^2} - \left[\frac{A-B-B(4\alpha+\beta)}{(4\alpha+\beta)}\right](A-B) \quad , if \mu \geq \frac{(4\alpha+\beta+1)(4\alpha+\beta)}{2(3\alpha+\beta)} & (26) \end{cases}$$

and the results are sharp.

**Proof:**

On Expanding (7) we have

$$\beta + (4a_2\alpha + a_2\beta)z + (6a_3\alpha + 2a_3\beta - 4a_2^2\alpha - a_2^2\beta)z^2 \prec 1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2)z^2 + \ldots \quad (27)$$

After identifying the terms in (27), we have

$$|a_3 - \mu a_2^2| \leq \left[\left|\frac{(A-B)(c_2-Bc_1^2)}{2(3\alpha+\beta)} + \frac{(A-B)^2c_1^2}{2(4\alpha+\beta)(3\alpha+\beta)} - \frac{(A-B)^2\mu}{(4\alpha+\beta)^2}c_1^2\right|\right]$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{2(3\alpha+\beta)} + \left[\left|\frac{(A-B)^2}{2(4\alpha+\beta)(3\alpha+\beta)} - \frac{B(A-B)}{2(3\alpha+\beta)} - \frac{(A-B)^2\mu}{(4\alpha+\beta)^2}\right| - \frac{A-B}{2(3\alpha+\beta)}\right]|c_1|^2 \quad (28)$$

**Case I :** If, $\mu \leq \left[\frac{(A-B)-B(4\alpha+\beta)}{2(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$, then

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(3\alpha+\beta)} + \frac{(A-B)}{2(3\alpha+\beta)}\left[\frac{(A-B)-B(4\alpha+\beta)-4\alpha-\beta}{(4\alpha+\beta)}\right] - \frac{(A-B)^2\mu}{(4\alpha+\beta)^2} \quad (29)$$

**Subcase I(a) :** If, $\mu \leq \left[\frac{(A-B)-B(4\alpha+\beta)-4\alpha-\beta}{2(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$, then

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{2(3\alpha+\beta)}\left[\frac{(A-B)-B(4\alpha+\beta)}{(4\alpha+\beta)}\right] - \frac{(A-B)^2\mu}{(4\alpha+\beta)^2} \quad (30)$$

**Subcase I(b) :** If, $\mu \geq \left[\frac{(A-B)-B(4\alpha+\beta)-4\alpha-\beta}{2(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$, then

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(3\alpha+\beta)} \quad (31)$$

**Case II :** If, $\mu \geq \left[\frac{(A-B)-B(4\alpha+\beta)}{2(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$, then

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{2(3\alpha+\beta)} + \frac{(A-B)^2\mu}{(4\alpha+\beta)^2} - \frac{A-B}{2(3\alpha+\beta)}\left\{\frac{(A-B)-B(4\alpha+\beta)+4\alpha+\beta}{(4\alpha+\beta)}\right\} \quad (32)$$

**Subcase II(a) :** If, $\mu \leq \left[\frac{(A-B)-B(4\alpha+\beta)+4\alpha+\beta}{2(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$, then

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(3\alpha+\beta)} \tag{33}$$

**Subcase II(b) :** If , $\mu \geq \left[\frac{(A-B)-B(4\alpha+\beta)+4\alpha+\beta}{2(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$,then

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2\mu}{(4\alpha+\beta)^2} - \left[\frac{(A-B)-B(4\alpha+\beta)}{2(3\alpha+\beta)(4\alpha+\beta)}\right](A-B) \tag{34}$$

Subcase II(a) and I(b) gives the common result.

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(3\alpha+\beta)} \tag{35}$$

Under the restriction

$$\frac{(4\alpha+\beta)[(A-B)-B(4\alpha+\beta)-4\alpha-\beta]}{2(3\alpha+\beta)(A-B)} \leq \mu \leq \frac{(4\alpha+\beta)[(A-B)-B(4\alpha+\beta)+4\alpha+\beta]}{2(3\alpha+\beta)(A-B)}$$

thiscomplets the theorem, hence the results are sharp.

**Extremal function**

Extreme value for first and third function is $z\{1+\mathrm{p}z\}^q$ (36)

where $\mathrm{p} = \frac{(A-B)(3\alpha+\beta)-(A-B)(4\alpha+\beta)+\mathrm{B}(4\alpha+\beta)^2}{(4\alpha+\beta)(3\alpha+\beta)}$

$$\mathrm{q} = \frac{(A-B)(3\alpha+\beta)}{(A-B)(3\alpha+\beta)-(A-B)(4\alpha+\beta)+\mathrm{B}(4\alpha+\beta)^2},$$

Extreme value for second function is $\frac{z}{(1-z^2)^{\frac{A-B}{2(3\alpha+\beta)}}}$ (37)

## Singularities:

Special cases (35)when A≠B

i) If A$>0$, $B>0$ then this inequality is holds only for A>B.

ii) If A$>0$, $B<0$ then this inequality is holds for all values of A and B.

iii)If A$<0$, $B<0$ then this inequality is holds only for B>A.

iv) If A$<0$, $B>0$ then this case does not valid

**THEOREM 3. :** Prove that

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{(4\alpha+\beta+2)\delta^2}{(4\alpha+\beta)(3\alpha+\beta)} - \frac{4\mu\delta^2}{(4\alpha+\beta)^2} \quad , if \mu \leq \frac{\{\delta(4\alpha+\beta+2)-4\alpha-\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)} & (38) \\ \frac{\delta}{3\alpha+\beta} \; , if \; \frac{\{\delta(4\alpha+\beta+2)-4\alpha-\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)} \leq \mu \leq \frac{\{\delta(4\alpha+\beta+2)+4\alpha+\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)} & (39) \\ \frac{4\mu\delta^2}{(4\alpha+\beta)^2} - \frac{(4\alpha+\beta+2)\delta^2}{(4\alpha+\beta)(3\alpha+\beta)} \quad , if \mu \geq \frac{\{\delta(4\alpha+\beta+2)+4\alpha+\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)} & (40) \end{cases}$$

and the results are sharp.

**Proof:**

On Expanding (8) we have

$$\beta + (4a_2\alpha + a_2\beta)z + (6a_3\alpha + 2a_3\beta - 4a_2^2\alpha - a_2^2\beta)z^2 \prec 1 + 2\delta c_1 z + 2\delta(c_2 + \delta c_1^2)z^2 + \ldots \tag{41}$$

After identifying the terms in (41), we have

$$|a_3 - \mu a_2^2| \leq \left| \frac{1}{(3\alpha+\beta)} \left\{ \delta c_2 + \delta^2 c_1^2 + \frac{2\delta^2 c_1^2}{4\alpha+\beta} \right\} - \mu \frac{4\delta^2 c_1^2}{(4\alpha+\beta)^2} \right|$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{\delta}{(3\alpha+\beta)} + \left[ \left| \frac{(4\alpha+\beta+2)\delta^2}{(4\alpha+\beta)} - \frac{4\delta^2\mu}{(4\alpha+\beta)^2} \right| - \frac{\delta}{(3\alpha+\beta)} \right] |c_1|^2 \tag{42}$$

**Case I :** If , $\mu \leq \frac{(4\alpha+\beta+2)(4\alpha+\beta)}{4(3\alpha+\beta)}$, then

$$|a_3 - \mu a_2^2| \leq \frac{\delta}{(3\alpha+\beta)} + \left[ \frac{(4\alpha+\beta+2)\delta^2}{(3\alpha+\beta)(4\alpha+\beta)} - \frac{\delta}{(3\alpha+\beta)} - \frac{4\delta^2\mu}{(4\alpha+\beta)^2} \right] |c_1|^2 \tag{43}$$

**Subcase I(a) :** If , $\mu \leq \frac{\{\delta(4\alpha+\beta+2)-4\alpha-\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)}$, then

$$|a_3 - \mu a_2^2| \leq \left[ \frac{(4\alpha+\beta+2)\delta^2}{(4\alpha+\beta)} - \frac{4\delta^2\mu}{(4\alpha+\beta)^2} \right] \tag{44}$$

**Subcase I(b) :** If , $\mu \geq \frac{\{\delta(4\alpha+\beta+2)-4\alpha-\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)}$, then

$$|a_3 - \mu a_2^2| \leq \frac{\delta}{(3\alpha+\beta)} \tag{45}$$

**Case II:** If , $\mu \geq \frac{(4\alpha+\beta+2)(4\alpha+\beta)}{4(3\alpha+\beta)}$, then

$$|a_3 - \mu a_2^2| \leq \frac{\delta}{(3\alpha+\beta)} + \left[ \frac{4\delta^2\mu}{(4\alpha+\beta)^2} - \frac{\delta\{\delta(4\alpha+\beta+2)+4\alpha+\beta\}}{4(3\alpha+\beta)(4\alpha+\beta)} \right] |c_1|^2 \tag{46}$$

**Subcase II(a) :** If , $\mu \leq \frac{\{\delta(4\alpha+\beta+2)+4\alpha+\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)}$, then

$$|a_3 - \mu a_2^2| \le \frac{\delta}{(3\alpha+\beta)} \quad (47)$$

**Subcase II(b) :** If, $\mu \ge \frac{\{\delta(4\alpha+\beta+2)+4\alpha+\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)}$,then

$$|a_3 - \mu a_2^2| \le \frac{4\delta^2\mu}{(4\alpha+\beta)^2} - \frac{(4\alpha+\beta+2)\delta^2}{(4\alpha+\beta)(3\alpha+\beta)} \quad (48)$$

Subcase II(a) and I(b) give the common result.

$$|a_3 - \mu a_2^2| \le \frac{\delta}{(3\alpha+\beta)} \quad (49)$$

Under the restriction

$$\frac{\{\delta(4\alpha+\beta+2)-4\alpha-\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)} \le \mu \le \frac{\{\delta(4\alpha+\beta+2)+4\alpha+\beta\}(4\alpha+\beta)}{4\delta(3\alpha+\beta)}$$

Thiscomplets the theorem, hence the results are sharp.

## Extremal function

Extreme value for first and third function is $z\{1+\mathrm{p}z\}^{\mathrm{Q}}$ (50)

where $\mathrm{P} = \frac{2\delta(3\alpha+\beta)-\delta(4\alpha+\beta)(4\alpha+\beta+2)}{(4\alpha+\beta)(3\alpha+\beta)}$ , $\mathrm{Q} = \frac{2\delta(3\alpha+\beta)}{2\delta(3\alpha+\beta)-\delta(4\alpha+\beta)(4\alpha+\beta+2)}$

Extreme value for second function is $\frac{z}{(1-z^2)^{\frac{\delta}{(3\alpha+\beta)}}}$ (51)

## Singularities:

Special cases for (49)

1. If $\delta > 0$ then the result is hold for all values of $\delta$.
2. If $\delta < 0$then the result is not valid.

Hence only (1) case is applicable on this theorem.

**THEOREM 4. :** Prove that

$$|a_3 - \mu a_2^2| \le$$

$$\begin{cases} \frac{(A-B)\delta}{2(3\alpha+\beta)}\left[\frac{(A-B)\delta-B(4\alpha+\beta)}{(4\alpha+\beta)}\right] - \frac{(A-B)^2\delta^2\mu}{(4\alpha+\beta)^2}, if \mu \le \left[\frac{(A-B)\delta-B(4\alpha+\beta)-4\alpha-\beta}{2\delta(A-B)(3\alpha+\beta)}\right](4\alpha+\beta) & (52) \\ \frac{(A-B)\delta}{2(3\alpha+\beta)}, if \frac{(4\alpha+\beta)[(A-B)\delta-B(4\alpha+\beta)-4\alpha-\beta]}{2\delta(3\alpha+\beta)(A-B)} \le \mu \le \frac{(4\alpha+\beta)[(A-B)\delta-B(4\alpha+\beta)+4\alpha+\beta]}{2\delta(3\alpha+\beta)(A-B)} & (53) \\ \frac{(A-B)^2\delta^2\mu}{(4\alpha+\beta)^2} - \left[\frac{(A-B)\delta-B(4\alpha+\beta)}{2(3\alpha+\beta)(4\alpha+\beta)}\right](A-B)\delta , if \mu \ge \frac{(4\alpha+\beta)[(A-B)\delta-B(4\alpha+\beta)+4\alpha+\beta]}{2\delta(3\alpha+\beta)(A-B)} & (54) \end{cases}$$

andthe results are sharp.

**Proof:**

On Expanding (9) we have

$$\beta + (4a_2\alpha + a_2\beta)z + (6a_3\alpha + 2a_3\beta - 4a_2^2\alpha - a_2^2\beta)z^2 \prec 1 + (A-B)c_1\delta z + (A-B)\delta(c_2 - B\delta c_1^2)z^2 + \dots \quad (55)$$

After identifying the terms in (55), we have

$$|a_3 - \mu a_2^2| \leq \left[\left|\frac{(A-B)\delta}{2(3\alpha+\beta)} + \left\{c_2 - B\delta c_1^2 + \frac{(A-B)^2\delta^2 c_1^2}{2(4\alpha+\beta)(3\alpha+\beta)}\right\} - -\frac{(A-B)^2\mu\delta^2}{(4\alpha+\beta)^2}c_1^2\right|\right]$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\delta}{2(3\alpha+\beta)} + \left[\left|\frac{(A-B)^2\delta^2}{2(4\alpha+\beta)(3\alpha+\beta)} - \frac{B(A-B)\delta}{2(3\alpha+\beta)} - \frac{(A-B)^2\delta^2\mu}{(4\alpha+\beta)^2}\right| - \frac{(A-B)\delta}{2(3\alpha+\beta)}\right]|c_1|^2 \quad (56)$$

**Case I :**If , $\mu \leq \left[\frac{(A-B)\delta - B(4\alpha+\beta)}{2\delta(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$,then

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\delta}{(3\alpha+\beta)} + \frac{(A-B)\delta}{2(3\alpha+\beta)}\left[\frac{(A-B)\delta - B(4\alpha+\beta) - 4\alpha - \beta}{(4\alpha+\beta)}\right] - \frac{(A-B)^2\delta^2\mu}{(4\alpha+\beta)^2} \quad (57)$$

**Subcase I(a) :** If , $\mu \leq \left[\frac{(A-B)\delta - B(4\alpha+\beta) - 4\alpha - \beta}{2\delta(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$,then

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\delta}{2(3\alpha+\beta)}\left[\frac{(A-B)\delta - B(4\alpha+\beta)}{(4\alpha+\beta)}\right] - \frac{(A-B)^2\delta^2\mu}{(4\alpha+\beta)^2} \quad (58)$$

**Subcase I(b) :** If , $\mu \geq \left[\frac{(A-B)\delta - B(4\alpha+\beta) - 4\alpha - \beta}{2\delta(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$,then

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\delta}{2\ (3\alpha+\beta)} \quad (59)$$

**Case II:** If ,$\mu \geq \left[\frac{(A-B)\delta - B(4\alpha+\beta)}{2\delta(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$,then

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\delta}{2(3\alpha+\beta)} + \frac{(A-B)^2\delta^2\mu}{(4\alpha+\beta)^2} - \frac{(A-B)\delta}{2(3\alpha+\beta)}\left\{\frac{(A-B)\delta - B(4\alpha+\beta) + 4\alpha + \beta}{(4\alpha+\beta)}\right\} \quad (60)$$

**Subcase II(a) :** If , $\mu \leq \left[\frac{(A-B)\delta - B(4\alpha+\beta) + 4\alpha + \beta}{2\delta(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$,then

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\delta}{2(3\alpha+\beta)} \quad (61)$$

**Subcase II(b) :** If , $\mu \geq \left[\frac{(A-B)\delta - B(4\alpha+\beta) + 4\alpha + \beta}{2\delta(A-B)(3\alpha+\beta)}\right](4\alpha+\beta)$,then

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2\delta^2\mu}{(4\alpha+\beta)^2} - \frac{(A-B)\delta}{2(3\alpha+\beta)}\left[\frac{(A-B)\delta - B(4\alpha+\beta)}{(4\alpha+\beta)}\right] \quad (62)$$

subcase II(a) and I(b) give the common result.

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\delta}{2(3\alpha+\beta)} \quad (63)$$

Under the restriction

$$\frac{(4\alpha+\beta)\left[(A-B)\delta-B(4\alpha+\beta)-4\alpha-\beta\right]}{2\delta(3\alpha+\beta)(A-B)}\leq\mu\leq\frac{(4\alpha+\beta)\left[(A-B)\delta-B(4\alpha+\beta)+4\alpha+\beta\right]}{2\delta(3\alpha+\beta)(A-B)}$$

Thiscomplets the theorem, hence the results are sharp.

**Extremal function**

Extreme value for first and third function is $z\{1+pz\}^{q}$ (64)

where $p=\frac{(A-B)\delta(3\alpha+\beta)-(A-B)\delta\,(4\alpha+\beta)+B(4\alpha+\beta)^2}{(4\alpha+\beta)(3\alpha+\beta)}$

$q=\frac{(A-B)\delta(3\alpha+\beta)}{(A-B)\delta(3\alpha+\beta)-(A-B)\,\delta(4\alpha+\beta)+B(4\alpha+\beta)^2}$ ,

Extreme value for second function is $\frac{z}{(1-z^2)^{\frac{(A-B)\delta}{2(3\alpha+\beta)}}}$ (65)

## Singularities:

Special casesfor (63) when $A\neq B$

i) In the case of $A>0, B>0, \delta>0$ or $A<0, B<0, \delta<0$ then ,this inequality holds good only for A>B.

ii) ) In the case of $A>0, B>0, \delta<0$ or $A<0, B<0, \delta>0$then, this inequality holds good only for B >A.

iii) ) In the case of $A>0, B<0, \delta<0$ or $A<0, B>0, \delta>0$ then, this inequality does not hold for all values of A and B.

iv) ) In the case of $A>0, B<0, \delta>0$ then, this inequality holds good for all values of A and B.

## 3. Concluding Remarks

If we take A = 1 and B = -1 in the result of theorem 2 , we get the result of theorem 1, therefore our result for the theorem 2 reduces to the result of the theorem1. Hence theorem 2 is the generalization of theorem 1. And the results are sharp and also if we put A = 1 and B = -1 in extremal function of theorem 2, we get the extremal function of theorem 1.

Similarly if we take A = 1 and B = -1in the result of theorem 4 , we get the result of theorem 3, therefore our result for the theorem 4 reduces to the result of the theorem 3. Hence theorem 4 is the generalization of theorem 3. And the results are sharp and also if we put A = 1 and B = -1 in extremal function of theorem 4, we get the extremal function of theorem 3.

The extremal function given by [(50) and (51)] increases as $\delta$ increases and decreases as $\delta$ decreases respectively and the extremal function given by [ (64) and (65)] also increases and decreases as $\delta$ increases and decreases respectively. Hence extremal function is an increasing function.

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## 4. References

[1] Al-Shaqsi K. and M.Darus, (2008).”On Fekete-Szegö problems for certain subclass of Analytic Functions” Applied Mathematical Sciences, vol. 2 no. 9-12,pp.431-441.

[2] G. S.Rathore, GurmeetSingh,LokendraKumawat, S. K.Gandhi,Preeti Kumawat (2019)”Some subclasses of a new class of analytic functions under Fekete-Szegö inequality” International journal of Research in Advent Technology, vol. 7, No. 1.

[3] GurmeetSingh,M. S. Saroaand B. S. Mehrok(2013)”Fekete-szegö inequality for a new class of analytic functions”, Elsevier, Proc. of International conference on Information and Mathematical Sciences, 90-93.

[4] H. R. Abdel-Gawad and D.K. Thomas (1992)”The Fekete-Szegö problem for Strongly close-to-convex functions” Proceedings of the Amercian Mathematical society, vol. 114, no.2 ,345-349.

[5] K Löwner,(1934).”Uber monotone Matrixfunktionen”,Math.Z38,177-216.

[6] K.O. Babalola.(2009).”The fifth and Sixth coefficient of $\alpha$– close- to –convex Function”,Kragujevac J. Math. 32, 5-12.

[7] L.Bieberbach, (1916).”Ubereinige Extremal Problemeim Gebiete der Konformen Abbildung” Math.77, 153-172.

[8] L.Bieberbach, (1916).Uberdie Koeffizientem derjenigem Potenzreihen, Welcheeine Schlithe Abbildung des EinheitskrisesVermittelen,Preuss. AKad.WissSitzungsb.940-955.

[9] M. H. Al-Abbadiand M.Darus (2011).Fekete-Sezegö theorem for a certain of analytic functions,SanisMalaysiana, vol. 40, no. 4, 385-389.

[10] M.Fekete, andG. Szego, (1933).”EineBemerkung Uberungerade Schlichte Funktionen”, J. London Math. Soc. 8, 85-89.

[11] P. N.Chhichra, (1977).”New Subclasses of the class of close to convex functions”,Procedure of American Mathematical Society ,62, 37-43.

[12] R.M. Goel and B. S.Mehrok(1982)”A Subclass of Univalent functions”, Houston Journal of Mathematics8, 343-357.

[13] R.M. Goel and B. S.Mehrok.(1990).”A coefficient inequality for certain classes of analytic function”,Tamkang Journal of Mathematics 22, 153-163.

[14] S.K. Gandhi, Gurmeet Singh, Preeti Kumawat, G, S. Rathore, Lokendra Kumawat (2018).“A new class of analytic functions with Fekete-Szegö inequality Using subordination Method”, International journal of Research in Advent Technology”, vol. 6, No. 9.

[15] S. R. Keogh, and E. R Merkes(1969)”A coefficient inequality for certain classes of analytic Functions”, Proceedings of the American Mathematical Society. Vol. 20, 8-12.

[16] S. R. Keogh, and E. R Merkes(1989).”A coefficient inequality for certain of analytic Functions”, Procedure of American Mathematical Society, 20, 8-12.

[17] T.Hayami, S. Owa, and H. M.Srivastava(2007).”Coefficient inequalities for certain class of analytic and univalent functions”, J. Ineq. Pure and Appl. Math.8(4).

[18] W.Kaplan, ,(1952).”Close –to-convex schlicht functions”, Michigan Mathematical Journal 1,169-185.